VI. *Short and easy Theorems for finding, in all cases, the Differences between the Values of Annuities payable Yearly, and of the same Annuities payable Half-yearly, Quarterly, or Momently. By the Rev.* Richard Price, *D. D. F. R. S. In a Letter to Sir* John Pringle, *Bart. P. R. S.*

R. Nov. 9, 1775.

THE values of annuities, as given in all the common tables, suppose them paid yearly. But it is well known, that generally they are paid half-yearly, and sometimes quarterly; and that this is a circumstance which always adds to their value. The difference between the values of annuities, according as they are paid in these different ways, I have seen no where stated with accuracy; and therefore, I have thought that the following attempt to do this may be of some use.

Annuities are of two sorts. They are either payable certainly or conditionally. Of the former sort are all annuities which are payable at fixed times, without depending on any contingency. Of the latter sort are all annuities on lives. I will first consider the first sort of annuities.

Let *r* denote the interest of 1 £. for a year; and *n* the term or number of years during which any anuuity is to be paid. Let P denote the value of the perpetuity, or the quotient arising from dividing 1 £. by its interest for a year. Let y denote the value of an annuity for n years, supposing it to be paid yearly; b its value, payable half-yearly; q its value, payable quarterly; and m its value, payable momently.

THEOREM I.

$$y = P - \frac{1}{r \times \overline{1+r}|^{n}}.$$

THEOREM II.

$$b = P - \frac{1}{r \times \overline{1+\frac{r}{2}}|^{2n}}.$$

THEOREM III.

$$q = P - \frac{1}{r \times \overline{1+\frac{r}{4}}|^{4n}}.$$

THEOREM IV.

$m = P - \frac{1}{rN}$. where N denotes the number which hath rn for its hyperbolic logarithm, and $rn \times 0.43429448$ for its logarithm in BRIGG's system.

EXAMPLE.

Let the rate of interest be 4 *per cent.* and the term 5 years, and consequently $r = 0,04$. $n = 5$. $P = 25$.

Then,

Then, $y=4.4518$
$h=4.4913$
$q=4.5120$
$m=4.5415$

EXAMPLE II.

Let the rate of intereſt be the ſame, and the term for which the annuity is payable 25 years.

Then, $y=15.6220$
$h=15.7118$
$q=15.7694$
$m=15.801$

EXAMPLE III.

Intereſt being the ſame, let the term be 50 years.

Then, $y=21.4822$
$h=21.5491$
$q=21.582$
$m=21.616$

EXAMPLE IV.

Intereſt being the ſame, let the term be 100 years.

Then, $y=24.505$
$h=24.523$
$q=24.532$
$m=24.542$

In the foregoing theorems it may be obſerved, that the *ratio* to one another, of the values of annuities payable yearly, half-yearly, quarterly, and momently, is greateſt when *n* is leaſt; that it decreaſes continually as *n* increaſes, till at laſt it vaniſhes, when *n* becomes infinite or the annuity is a perpetuity. Agreeably to this it appears, in the examples I have given, that the values in the firſt example differ more from one another in proportion than the values in the ſecond example; and that theſe alſo differ more than the values in the third; and that in the laſt example all the values are nearly the ſame.

Theſe values computed by Mr. DE MOIVRE's rules in his Treatiſe on Life-annuities, p. 86. and 124, &c. come out greater when *n* exceeds, and leſs when *n* falls ſhort of 15 or 20 years. But thoſe rules ſuppoſe the half-yearly and quarterly intereſts of money to be leſs than half or a quarter of the yearly intereſt. For inſtance; the value of an annuity of 1£. payable half-yearly and quarterly for 50 years is, according to Mr. DE MOIVRE's rules, 21,699 and 21,772, or a 99th part and 74th part more than the value of the ſame annuity payable yearly, ſuppoſing money improved at 4 *per cent.* when the annuity is paid yearly; and at £. 1,98 *per cent.* when it is paid half-yearly; and at 0,985 *per cent.* when it is paid quarterly: That is; ſuppoſing money improved at a rate of half-yearly or quarterly intereſt, which, inſtead of being a half or a quarter of the yearly intereſt, is only that half-

half-yearly or quarterly payment which, in confequence of being laid up and improved at compound intereft, will in a year amount to the fum that makes the yearly intereft. It is obvious that this cannot be the proper method of computing thefe values. But not to infift on this; I will next ftate the different values of the fecond fort of annuities; or of life-annuities, according as they are fuppofed to be payable yearly, half-yearly, quarterly, or momently.

Let r as before be the intereft of 1 £. for a year; n the complement of a given life[(a)]; y, h, q, and m, the values refpectively of an annuity certain for n years payable yearly, half-yearly, quarterly, or momently; P the perpetuity; Y the prefent value of an annuity on a life whofe complement is n, payable yearly; H the value of the fame annuity payable half-yearly; and Q and M the values of the fame annuity payable quarterly and momently.

(*a*) The complement of a life is, in Mr. DE MOIVRE's hypothefis, the number of years it wants of 86. In all other cafes, it is double the expectation of a life; that is, it is double the quotient (diminifhed by $\frac{1}{2}$ unity) arifing from dividing the fum of all the living in a table of obfervations from the age (inclufive) of the given life to the extremity of life, by the number of the living at that age. See Effay I. in my Treatife on Reverfionary Payments.

 Then,

Then, $Y = P - \frac{1+r}{rn} \times y.$

$$H = P - \frac{1+\frac{r}{2}}{nr} \times h.$$

$$Q = P - \frac{1+\frac{r}{4}}{nr} \times q.$$

$$M = P - \frac{m}{nr}.$$

EXAMPLE I.

Let the life be ſuppoſed of the age of 36. The complement of ſuch a life is 50, according to Mr. DE MOIVRE's hypotheſis; and alſo very nearly, according to the Breſlaw and the Northampton tables of obſervations. Therefore, n will be 50. Let the rate of intereſt be 4 *per cent*, or $r = 0{,}04$. $P = 25$. $y = 21{,}482$. $h = 21{,}549$. $q = 21{,}582$. $m = 21{,}616$. See p. 111.

Therefore, $Y = 25 - \frac{1{,}04}{50 \times 0{,}04} \times 21{,}482 = 13{,}829$

$$H = 25 - \frac{1{,}02}{50 \times 0{,}04} \times 21{,}549 = 14{,}010$$

$$Q = 25 - \frac{1{,}01}{50 \times 0{,}04} \times 21{,}582 = 14{,}101$$

$$M = 25 - \frac{21{,}616}{50 \times 0{,}04} = 14{,}191$$

EXAMPLE II.

Let the life be ſuppoſed of the age of 61. The complement of this life is 25 by Mr. DE MOIVRE's hypothe-

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fis and the Northampton table of obfervations. Therefore, intereft fuppofed at 4 *per cent.*

$$Y=25-\frac{1,04}{25\times 0,04}\times 15,622=8,753$$

$$H=25-\frac{1,02}{25\times 0,04}\times 15,712=8,973$$

$$Q=25-\frac{1,01}{25\times 0,04}\times 15,769=9,072$$

$$M=25-\frac{15,801}{25\times 0,04}=9,199$$

The different values, given by thefe theorems, of life-annuities payable yearly, half-yearly, and quarterly, fuppofe nothing to be due to an annuitant for that year, half-year, or quarter, in which he fhall happen to die. If, on the contrary, he is to be entitled to fuch part of the annuity as fhall be proportioned to the time which fhall happen to intervene between his death and the time when the payment immediately preceding his death became due; or in other words, if the annuity is an annuity fecured by land, $\frac{y}{2n}$ muft be added to the firft theorem in order to obtain the value of fuch an annuity payable yearly. And in like manner, $\frac{h}{4n}$ muft be added to the fecond theorem to obtain the value of the fame annuity payable half-yearly: and $\frac{q}{8n}$ to the third theorem, to obtain its value payable quarterly.

The value, therefore, in the firft example, of an annuity payable yearly on a life aged 36 being 13,829; its value, if fecured by land, or to be enjoyed to the laft moment of

 life.

life, will be $13,829+\frac{21,482}{100}=14,043$. If ſecured by land and payable half-yearly, its value will be $14,010+\frac{21,549}{200}=14,117$. If ſecured by land and payable quarterly, its value will be $14,101+\frac{21,582}{400}=14,155$. The like values in the ſecond example are 9,065, 9,130, and 9,151.

Life-annuities payable monthly or weekly may be conſidered as of the ſame value with annuities payable momently; and it is evident, that they muſt be enjoyed nearly to the laſt moment of life.

From theſe rules and examples it may be gathered, that the difference between the values of annuities on lives payable yearly, half-yearly, quarterly, and momently, increaſes continually with the ages; but, if not ſecured by land, this difference can never be ſo great as a quarter of a year's purchaſe in the caſe of annuities payable yearly and half-yearly; three-eighths of a year's purchaſe in the caſe of annuities payable yearly and quarterly; and half a year's purchaſe in the caſe of annuities payable yearly and momently.

Mr. SIMPSON, in his Treatiſe on the Doctrine of Life-annuities, p. 78. and in his Select Exerciſes, p. 283. hath given a quarter of a year's purchaſe as the addition always to be made to the value of a life-annuity payable yearly, in order to obtain its value payable half-yearly; and three-eighths of a year's purchaſe, if its value payable quarterly is required. But it appears, that theſe are too large additions; and, whatever be the rate of intereſt, or the number of lives, a fifth of a year's purchaſe will be generally more than a ſufficient addition, if the value of the annuity is deſired payable half-yearly; and three-tenths of a year's purchaſe, if the value of the annuity is deſired payable quarterly. Mr. DE MOIVRE's rules, in p. 85 of his Book on Life-annuities, for finding the values of life-annuities payable half-yearly and quarterly from their values payable yearly, are ſtill leſs correct; for they ſuppoſe the difference between theſe values the ſame, whether the annuities are life-annuities, or annuities certain.

Mr. DODSON, in the firſt queſtion in the third volume of his Mathematical Repoſitory, hath given a rule for finding the value of an annuity ſecured by land and payable yearly, which coincides with that here given; and Mr. DE MOIVRE, in p. 338. of his Treatiſe on the Doctrine of Chances, hath given a theorem for this purpoſe, which alſo brings out nearly the ſame anſwers. But Mr. SIMPSON, in prob. I. p. 323. of his Select Exerciſes, makes the exceſs of the value of ſuch an annuity above the value of an annuity payable yearly, but not ſecured by land, double to the ſame exceſs derived from Mr. DODSON's and Mr. DE MOIVRE's rules. The truth is, that Mr. DODSON's rule gives the exact value; and that Mr. SIMPSON's problem gives the value, not of an annuity ſecured by land and payable yearly, but of an annuity ſecured by land and payable momently; and alſo, that his method of ſolution implies a rate of intereſt ſomewhat leſs when the annuity is payable momently than when it is payable yearly.

But

But to prevent all perplexity on this subject, I will subjoin the following investigations, which will be easily understood by those who are acquainted with the common methods of calculating the values of life-annuities.

Let r, as before, be the interest of £.1 for a year. Then the present value of £.1 payable at the end of one year, two years, three years, &c. will be $\frac{1}{1+r}$, $\frac{1}{\overline{1+r}|^2}$, $\frac{1}{\overline{1+r}|^3}$, &c. respectively. And the present value of an annuity certain for n years payable yearly is the sum of this series continued to n terms (b), or $\frac{1}{r}-\frac{1}{r\times\overline{1+r}|^n}=\text{P}-\frac{1}{r\times\overline{1+r}|^n}=y$.

In like manner, the present value of half £.1 (that is of 10 *s.*=£.0,5) payable at the end of half a year, a year, a year and a half, &c. reckoning half-yearly interest at half the annual interest, is $\frac{0,5}{1+\frac{r}{2}}$, $\frac{0,5}{\overline{1+\frac{r}{2}}|^2}$, $\frac{0,5}{\overline{1+\frac{r}{2}}|^3}$, &c. And the present value of an annuity certain payable half-yearly for n years, each payment to be half the yearly payment, is the sum of this series continued to $2n$ terms; or,

$$\frac{0,5}{\frac{r}{2}}-\frac{0,5}{\frac{r}{2}\times\overline{1+\frac{r}{2}}|^{2n}}=\frac{1}{r}-\frac{1}{r\times\overline{1+\frac{r}{2}}|^{2n}}=\text{P}-\frac{1}{r\times\overline{1+\frac{r}{2}}|^{2n}}=b.$$

(b) In the postscript it will be proved, that the sum of n terms of the series $\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}$, &c. is $\frac{1}{a-1}-\frac{1}{a^n\times\overline{a-1}}$. Substitute $1+r$ for a, and it will appear, that the sum of n terms of the series $\frac{1}{1+r}+\frac{1}{\overline{1+r}|^2}+\frac{1}{\overline{1+r}|^3}$, &c. is $\frac{1}{r}-\frac{1}{r\times\overline{1+r}|^n}$.

By

By the ſame ſteps it will appear, that the preſent value of an annuity certain for n years to be received in quarterly payments, each a quarter of the annual payment, is $\frac{0{,}25}{\frac{1}{4}r} - \frac{0{,}25}{\frac{1}{4}r \times \overline{1+\frac{r}{4}}^{\,4n}} = \text{P} - \frac{1}{r \times \overline{1+\frac{r}{4}}^{\,4n}} = q$. And alſo, that the preſent value of an annuity certain for n years, to be received in momently payments, each the ſame proportional part of the yearly payment that the moment is of the year, muſt be $\text{P} - \frac{1}{r \times \overline{1+\frac{r}{1000,\,\&c.}}^{\,1000,\,\&c.\,n}}$. But, by the binomial theorem, $\overline{1+\frac{r}{1000,\,\&c.}}^{\,1000,\,\&c.\,n} = 1 + rn + \frac{r^2n^2}{2} + \frac{r^3n^3}{2 \times 3} + \frac{r^4n^4}{2 \times 3 \times 4}$, &c. which ſeries approximates indefinitely to that number of which rn is the hyperbolic logarithm, by prob. 1. ſect. XI. vol. II. of Mr. SIMPSON's Fluxions; or by prop. 1. p. 40. of his Treatiſe on Trigonometry. Therefore, $\text{P} - \frac{1}{r \times \overline{1+\frac{1}{1000,\,\&c.}}^{\,1000,\,\&c.\,n}} = \text{P} - \frac{1}{r\text{N}} = \text{M}$, as explained before. See p. 110.

If the value of an annuity of £. 1 for n years is required payable half-yearly, and the half-yearly intereſt of £. 1, inſtead of being half the yearly intereſt (or $\frac{r}{2}$), is ſuppoſed to be $\overline{1+r}^{\,\frac{1}{2}} - 1$; the anſwer will be $\frac{0{,}5}{\overline{1+r}^{\,\frac{1}{2}}} + \frac{0{,}5}{1+r} + \frac{0{,}5}{\overline{1+r}^{\,1\frac{1}{2}}} + \frac{0{,}5}{\overline{1+r}^{\,2}}$, &c. continued to $2n$ terms =

=

$=\frac{0,5}{\overline{1+r}|^{\frac{1}{2}}-1}-\frac{0,5}{\overline{1+r}|^{n}\times\overline{1\times r}|^{\frac{1}{2}}-1}=\overline{1-\frac{1}{\overline{1+r}|^{n}}}\times\frac{1}{2\times\overline{1+}|^{\frac{1}{2}}-2}$; which value is to $\overline{1-\frac{1}{\overline{1+r}|^{n}}}\times\frac{1}{r}$ (the value of the ſame annuity payable yearly ſuppoſing the yearly intereſt of £. 1 to be r) as $\frac{\frac{1}{2}}{\overline{1+r}|^{\frac{1}{2}}-1}$ to $\frac{1}{r}$, agreeably to Mr. DE MOIVRE's deduction in his Treatiſe on Life-annuities, p. 125. fourth edit. This implying, in the caſe of annuities payable half-yearly, a ſmaller intereſt than half the yearly intereſt (for $\overline{1+r}|^{\frac{1}{2}}-1$ is leſs than $\frac{r}{2}$) gives the difference between their value and the value of annuities payable yearly, greater than the truth.

But to return to the inveſtigation of the theorems in the former part this paper.

Let us again call P the perpetuity, and y the value of an annuity certain for n years and payable yearly; it is well known that the value of £. 1 payable yearly on a life whoſe complement is n is (ſuppoſing an equal decrement of life) $\frac{n-1}{n\times\overline{1+r}}+\frac{n-2}{n\times\overline{1+r}|^{2}}+\frac{n-3}{n\times\overline{1+r}|^{3}}$, &c. continued to n terms(c) $=\text{P}-\frac{1+r}{nr}\times y=\text{Y}$.

In

(c) See Mr. DE MOIVRE's Treatiſe on Life-annuities, p. 99. fourth edit. Or his Doctrine of Chances, p. 311. third edition. Or Mr. DODSON's Mathematical Repoſitory, vol. II. p. 137. Or Mr. SIMPSON on Annuities and Reverſions, p. 14. In conſulting theſe writers, care ſhould be taken to remember, that they uſe r to denote the principal and intereſt of £. 1 for a year; whereas it hath been

In like manner, ſuppoſing money improved at an half-yearly intereſt equal to half the yearly intereſt, or to $\frac{r}{2}$, the value of the ſame annuity payable half-

been moſt convenient for me in theſe obſervations to make r ſtand only for the intereſt. In theſe writers, therefore, r ſignifies the ſame with $1+r$ in this paper; and $r-1$ the ſame with r.

It is ſaid above, that the value of an annuity payable yearly, on a life whoſe complement is n, is $\frac{n-1}{n\times\overline{1+r}}+\frac{n-2}{n\times\overline{1+r}|^{2}}+\frac{n-3}{n\times\overline{1+r}|^{3}}$, &c. continued to n terms. This expreſſion is equal to $\frac{n}{n\times\overline{1+r}}+\frac{n}{n\times\overline{1+r}|^{2}}+\frac{n}{n\times\overline{1+r}|^{3}}$, &c. ($n$) $-\frac{1}{n}\times\overline{\frac{1}{1+r}+\frac{2}{\overline{1+r}|^{2}}+\frac{3}{\overline{1+r}|^{3}}}$, &c. ($n$). But $\frac{n}{n\times\overline{1+r}}+\frac{n}{n\times\overline{1+r}|^{2}}+\frac{n}{n\times\overline{1+r}|^{3}}$, &c. $\left(=\frac{1}{1+r}+\frac{1}{\overline{1+r}|^{2}}+\frac{1}{\overline{1+r}|^{3}}\right.$, &c.$\left.\right)=\frac{1}{r}-\frac{1}{r\times\overline{1+r}|^{n}}=y$ (ſee p. 118.) Alſo, by a theorem which will be demonſtrated in the poſtſcript, and putting a for any given quantity, $\frac{1}{a}+\frac{2}{a^{2}}+\frac{3}{a^{3}}$, &c. continued to n terms $=\frac{a}{\overline{a-1}|^{2}}-\frac{n}{a^{n}}\times\frac{1}{a-1}-\frac{1}{a^{n}}\times\frac{a}{\overline{a-1}|^{2}}$. Therefore, if $1+r$ is ſubſtituted for a, and y for $\frac{1}{r}-\frac{1}{r\times\overline{1+r}|^{n}}$, the ſum (multiplied by $\frac{1}{n}$) of n terms of the ſeries $\frac{1}{1+r}+\frac{2}{\overline{1+r}|^{2}}+\frac{3}{\overline{1+r}|^{3}}$, &c. will come out $\frac{1+r}{nr}\times y-\frac{1}{r}\times\frac{1}{\overline{1+r}|^{n}}$; or $\frac{1+r}{nr}\times y+y-\frac{1}{r}$. Therefore, the ſeries $\frac{1}{n}\times\overline{\frac{1}{1+r}+\frac{2}{\overline{1+r}|^{2}}+\frac{3}{\overline{1+r}|^{3}}}$, &c. continued to n terms and ſubtracted from the ſeries $\frac{1}{1+r}+\frac{1}{\overline{1+r}|^{2}}+\frac{1}{\overline{1+r}|^{3}}$, &c. continued to n terms; that is, the value of the life will be $y-\overline{\frac{1+r}{nr}\times y+y-\frac{1}{r}}=\frac{1}{r}-\frac{1+r}{nr}\times y=P-\frac{1+r}{nr}\times y=Y$.

yearly, is $\frac{1}{2}\times\overline{\frac{n-\frac{1}{2}}{n\times\left(1+\frac{r}{2}\right)}+\frac{n-1}{n\times\left(1+\frac{r}{2}\right)^{2}}+\frac{n-\frac{3}{2}}{n\times\left(1+\frac{r}{2}\right)^{3}}}$, &c. continued to $2n$ terms $=\frac{1}{2}\times\overline{\frac{n}{n\times\left(1+\frac{r}{2}\right)}+\frac{n}{n\times\left(1+\frac{r}{2}\right)^{2}}+\frac{n}{n\times\left(1+\frac{r}{2}\right)^{3}}}$, &c. continued to $2n$ terms $-\frac{1}{2}\times\overline{\frac{\frac{1}{2}}{n\times\left(1+\frac{r}{2}\right)}+\frac{1}{n\times\left(1+\frac{r}{2}\right)^{2}}+\frac{\frac{3}{2}}{n\times\left(1+\frac{r}{2}\right)^{3}}}$, &c. continued to $2n$ terms. But the ſum of the firſt of theſe two ſeries, or of $\frac{1}{2}\times\frac{n}{n\times\left(1+\frac{r}{2}\right)}+\frac{n}{n\times\left(1+\frac{r}{2}\right)^{2}}$, &c. $\left(=\frac{1}{2}\times\frac{1}{1+\frac{r}{2}}+\frac{1}{\left(1+\frac{r}{2}\right)^{2}}\right.$, &c.) is b, ſee p. 118. And the ſum of the ſecond ſeries is the ſame with half the ſum of the ſeries $\frac{1}{2n}\times\overline{\frac{1}{1+\frac{r}{2}}+\frac{2}{\left(1+\frac{r}{2}\right)^{2}}+\frac{3}{\left(1+\frac{r}{2}\right)^{3}}}$, &c. $(2n)$. But by the theorem mentioned in the laſt note, the ſum of n terms of the ſeries $\frac{1}{a}+\frac{2}{a^{2}}+\frac{3}{a^{3}}$, &c. is $\frac{a}{(a-1)^{2}}-\frac{n}{a^{n}}\times\frac{1}{a-1}-\frac{1}{a^{n}}\times\frac{a}{(a-1)^{2}}$. Therefore, if $1+\frac{r}{2}$ is ſubſtituted for a, $2n$ for n, and b for $\frac{1}{r}-\frac{1}{r\times\left(1+\frac{r}{2}\right)^{2n}}$, the ſum of the ſecond ſeries (that is, of $\frac{1}{2}\times\frac{1}{2n}\times\overline{\frac{1}{1+\frac{r}{2}}+\frac{2}{\left(1+\frac{r}{2}\right)^{2}}+\frac{3}{\left(1+\frac{r}{2}\right)^{3}}}$, &c. $(2n)$) will come out $\frac{1+\frac{r}{2}}{nr}\times b-\frac{1}{r}\times\frac{1}{\left(1+\frac{r}{n}\right)^{2n}}$, or $\frac{1+\frac{r}{2}}{nr}\times b+b-\frac{1}{r}$.

Therefore,

Therefore, the ſecond ſeries ſubtracted from the firſt, leaves $\frac{1}{r} - \frac{1+\frac{r}{2}}{nr} \times h = P - \frac{1+\frac{r}{2}}{nr} \times h = H$, agreeably to the ſecond theorem in p. 114.

By reaſoning in the ſame way it may be eaſily found, that $Q = P - \frac{1+\frac{r}{4}}{nr} \times q$; and $M = P - \frac{1+\frac{r}{1000, \&c.}}{nr} \times m = P - \frac{m}{nr}$, agreeably to the third and fourth theorems in p. 114.

Theſe theorems, I have ſaid, ſuppoſe that an annuitant is entitled to no payment for that year, half-year, or quarter, in which he dies. If, on the contrary, he is to be intitled, when he dies, to ſuch a part of the yearly, half-yearly, or quarterly payment as ſhall bear the ſame proportion to the ſaid payments reſpectively, as the intermediate time between the laſt payment and his death bears to the whole year, half-year, or quarter; in this caſe, ſuppoſing the annuity payable yearly, it is evident, ſince there is the ſame chance for his dying in one half of any year as in the other, that he will have an expectation of half a year's payment more than he would be otherwiſe intitled to. But the value of half £. 1 to be paid at the death of a perſon whoſe complement of life is n, is $\frac{1}{2} \times \frac{1}{n \times \overline{1+r}} + \frac{1}{2} \times \frac{1}{n \times \overline{1+r}|^2} + \frac{1}{2} \times \frac{1}{n \times \overline{1+r}|^3}$, &c. continued to n terms (d) $= \frac{y}{2n}$.

(d) P. 118.

In like manner, a perſon who enjoys an annuity ſecured by land, payable half-yearly, will have an expectation of a quarter of a year's payment more than he could be otherwiſe intitled to; the value of which is $\frac{1}{4n} \times \frac{1}{1+\frac{r}{2}} + \frac{1}{\overline{1+\frac{r}{2}}|^2} + \frac{1}{\overline{1+\frac{r}{2}}|^3}$, &c. continued to $2n$ terms $= \frac{h}{4n}$.

By the ſame reaſoning it will appear, that $\frac{q}{8n}$ is the addition to be made to the value of an annuity payable quarterly, in order to obtain its value when ſecured by land.

P O S T S C R I P T.

IN the note p. 121. the expreſſion $\frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$ is given as the ſum of n terms of the ſeries $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^5}$, &c. to $\frac{1}{a^n}$, and the expreſſion $\frac{a}{\overline{a-1}|^2} - \frac{n}{a^n} \times \frac{1}{a-1} - \frac{1}{a^n} \times \frac{a}{\overline{a-1}|^2}$, is given as the ſum of n terms of the ſeries $\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4}$, &c.

The following inveſtigation of theſe theorems being very eaſy, will not, perhaps, be unacceptable to thoſe who have ſtudied this ſubject.

Put $A = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4}$, &c. $\frac{1}{a^n}$. $B = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4}$, &c. $\frac{n}{a^n}$.

Then $A \times a = 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}}$,

and $A \times a - 1 + \frac{1}{a^n} = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}} + \frac{1}{a^n} = A$,

and

and $A \times a - A (= A \times \overline{a-1}) = 1 - \frac{1}{a^n}$.

Therefore, $A = \frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$, which is the firſt theorem.

Again, $A \times a = 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, &c. to $\frac{1}{a^{n-1}}$,

and $B \times a = 1 + \frac{2}{a} + \frac{3}{a^2} + \frac{4}{a^3}$, &c. to $\frac{n}{a^{n-1}}$.

Therefore, $B \times a - A \times a = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3}$, &c. to $\frac{n-1}{a^{n-1}}$.

To both ſides of the laſt equation add $\frac{n}{a^n}$, and it will appear, that

$$B \times a - A \times a + \frac{n}{a^n} = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4} +, \text{ \&c. to } \frac{n-1}{a^{n-1}} + \frac{n}{a^n} = B.$$

Therefore, $B \times a - B = B \times \overline{a-1} = A \times a - \frac{n}{a^n}$; and $B = \frac{A \times a}{a-1} - \frac{n}{a^{n+1} - a^n}$.

For A, in this laſt equation, ſubſtitute its equal, or $\frac{1}{a-1} - \frac{1}{a^n} \times \frac{1}{a-1}$, and the reſulting equation will be $\frac{a}{\overline{a-1}|^2} - \frac{n}{a^n} \times \frac{1}{a-1} - \frac{1}{a^n} \times \frac{a}{\overline{a-1}|^2} = B$, which is the ſecond theorem.

When n is infinite, all but the firſt terms in both theſe theorems vaniſh; and therefore, $\frac{1}{a-1}$ is the ſum of the ſeries $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, &c. continued infinitely; and $\frac{a}{\overline{a-1}|^2}$ is the ſum of the ſeries $\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3}$, &c. continued infinitely.

By

By a like deduction, putting

$$C=\frac{1}{a}+\frac{2\times 2}{a^2}+\frac{3\times 3}{a^3}+\frac{4\times 4}{a^4}, \text{ \&c. to } \frac{n^2}{a^n},$$

and $D=\frac{1}{a}+\frac{2\times 2\times 2}{a^2}+\frac{3\times 3\times 3}{a^3}+\frac{4\times 4\times 4}{a^4}$, &c. to $\frac{n^3}{a^n}$, it may be found that $C=\frac{A+2B+1}{a-1}-\frac{\overline{n+1}|^2}{a^{n+1}-a^n}$, and $D=\frac{A+3B+3C+1}{a-1}-\frac{\overline{n+1}|^3}{a^{n+1}-a^n}$. And consequently, substituting the values of A and B, that

$$C=\frac{a^2+a}{\overline{a-1}|^3}-\frac{n^2}{a^n}\times\frac{1}{a-1}-\frac{2an}{a^n}\times\frac{1}{\overline{a-1}|^2}-\frac{a^2+a}{a^n}\times\frac{1}{\overline{a-1}|^3}.$$

And, substituting the values of A, B, C, that

$$D=\frac{a^3+4a^2+a}{\overline{a-1}|^4}-\frac{n^3}{a^n}\times\frac{1}{a-1}-\frac{3an^2}{a^n}\times\frac{1}{\overline{a-1}|^2}-\frac{3a^2n+3an}{a^n}\times\frac{1}{\overline{a-1}|^3}-\frac{a^3+4a^2+a}{a^n}\times\frac{1}{\overline{a-1}|^4}.$$

Or, since all but the first terms in these expressions vanish when n is infinite, that the sum of the series $\frac{1}{a}+\frac{4}{r}+\frac{9}{a^3}$, &c. continued infinitely is $\frac{a^2+a}{\overline{a-1}|^3}$; and that the sum of the series $\frac{1}{a}+\frac{8}{a^2}+\frac{27}{a^3}+\frac{64}{a^4}$, &c. continued infinitely is $\frac{a^3+4a^2+a}{\overline{a-1}|^4}$.

These are all the theorems necessary for calculating the values of annuities on single lives, and on any two or three joint lives, upon the hypothesis of an equal decrement of life.

Supposing r the interest of £. 1 for a year, the sum of n terms of the series $\frac{1}{1+r}+\frac{1}{\overline{1+r}|^2}+\frac{1}{\overline{1+r}|^3}$, &c. is the present value of an annuity certain for n years; and

$\frac{1}{1+r}$

$\frac{1}{1+r}+\frac{2}{\overline{1+r}|^2}+\frac{3}{\overline{1+r}|^3}+\frac{4}{\overline{1+r}|^4}$, (continued to n terms) is the present value of an annuity certain beginning with £. 1, and increasing to £. 2 the second year, to £. 3 the third year, &c.

If this last annuity is not an annuity certain for a given term, but a life-annuity, the value of it (supposing n the complement of life, A the value of an annuity certain for n years, G the value of two equal joint lives whose common complement is n, P the perpetuity, and p the value of £. 1 to be received at the end of n years) will be $\overline{A-G}\times n+n\times p\times P-A\times P\times\overline{1+r}$.

EXAMPLES.

Let the term be forty-one years, and the rate of interest 4 per cent.

The value of an annuity of £. 1 certain for this term is £. 20.

The value of an annuity certain for the same term, and beginning with £. 1 at the end of the first year, but increasing to £. 2 at the end of the second year, to £. 3 at the end of the third year, and so on till it becomes £. 41 at the end of the forty-first year, is (by the second theorem, putting $1+r$, or 1,04 for a) £. 214 10*s.*

The value of an annuity increasing at this rate without end is £. 650.

If

If the annuity is a life-annuity which is to increaſe at the rate of £. 1 every year during the whole poſſible continuance of a life whoſe complement is forty-one years (or whoſe age is about forty-five) the preſent value of it will be, by the laſt theorem, £. 133 14*s.*; taking the probability of the duration of human life according to Mr. DE MOIVRE's hypotheſis; which agrees nearly with Dr. HALLEY's Table of Obſervations.